Recall:

For θ a Grassmann number, we have

$$\int d\theta = 0 \quad , \quad \int d\theta\, \theta = 1$$

For n generators $\{\theta_1, \dots, \theta_n\}$ we have

$$\{\theta_i, \theta_j\} = 0 \qquad \forall\ i, j$$

→ set of all linear combinations of $\{\theta_i\}$ with c-number coefficients

commuting

forms a "Grassmann algebra", called Λ^n.

→ arbitrary elements of Λ^n are expanded as

$$f(\theta) = f_0 + \sum_{i=1}^{n} f_i \theta_i + \sum_{i<j} f_{ij} \theta_i \theta_j + \cdots$$

$$= \sum_{0 \le k \le n} \frac{1}{k!} \sum_{\{i\}} f_{i_1 \dots i_k} \theta_{i_1} \cdots \theta_{i_k} \, ,$$

where $f_0, f_i, f_{ij}, \dots$ and $f_{i_1 \dots i_k}$ are anti-sym. c-numbers

also have: $f(\theta) = \sum_{k_i = 0,1} \tilde{f}_{k_1 \dots k_n} \theta_1^{k_1} \dots \theta_n^{k_n}$

Example: $n=2$

$$f(\theta) = f_0 + f_1 \theta_1 + f_2 \theta_2 + f_{12} \theta_1 \theta_2$$

$$= \tilde{f}_{00} + \tilde{f}_{10} \theta_1 + \tilde{f}_{01} \theta_2 + \tilde{f}_{11} \theta_1 \theta_2$$

The subset of Λ^n generated by monomials of even (resp. odd) power in θ_k is denoted by Λ_+^n (Λ_-^n):

$$\Lambda^n = \Lambda_+^n \oplus \Lambda_-^n$$

$\longrightarrow$ $\mathbb{Z}_2$-grading

<u>Note</u>: $\dim \Lambda^n = 2^n$, $\dim \Lambda_+^n = \dim \Lambda_-^n = 2^{n-1}$

We have:

$$\theta_k^2 = 0$$

$$\theta_{k_1}\theta_{k_2}\cdots\theta_{k_n} = \varepsilon_{k_1 k_2 \cdots k_n}\, \theta_1 \theta_2 \cdots \theta_n$$

$$\theta_{k_1}\theta_{k_2}\cdots\theta_{k_m} = 0 \qquad (m > n),$$

where

$$\varepsilon_{k_1\cdots k_n} = \begin{cases} +1 & \text{if } \{k_1,\dots,k_n\} \text{ is even perm. of } \{1\cdots n\} \\ -1 & \text{if } \{k_1,\dots,k_n\} \text{ is odd perm. of } \{1,\dots,n\} \\ 0 & \text{otherwise} \end{cases}$$

<u>Differentiation</u>:

i) A differential operator acts on a function from left:

$$\frac{\partial\theta_j}{\partial\theta_i} = \frac{\partial}{\partial\theta_i}\theta_j = \delta_{ij}$$

ii) diff. operator anti-commutes with θ_k

$\longrightarrow$ Leibnitz rule gives

$$\frac{\partial}{\partial\theta_i}\left(\theta_j \cdot \theta_k\right) = \frac{\partial\theta_j}{\partial\theta_i}\,\theta_k - \theta_j \cdot \frac{\partial\theta_k}{\partial\theta_i} = \delta_{ij}\,\theta_k - \delta_{ik}\,\theta_j$$

iii) $$\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_j} + \frac{\partial}{\partial\theta_j}\frac{\partial}{\partial\theta_i} = 0 \qquad \text{(exercise)}$$

iv) $$\frac{\partial}{\partial\theta_i}\,\theta_j + \theta_j\,\frac{\partial}{\partial\theta_i} = \delta_{ij}$$

Relation between integration and differentiation

The following axioms hold for I denoting integration and D differentiation:

(1) $ID = 0$,

(2) $DI = 0$,

(3) $D(A) = 0 \Rightarrow I(BA) = I(B)A$

$\longrightarrow$ integration is same as differentiation:

$$\int d\theta_1\, d\theta_2 \cdots d\theta_n\, f(\theta_1, \theta_2, \dots, \theta_n)$$
$$= \frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2}\cdots\frac{\partial}{\partial\theta_n}\, f(\theta_1, \theta_2, \dots, \theta_n)$$

Change of integration variables

Let us consider $n=1$ first:

under a change $\theta' = a\theta$ ($a \in \mathbb{C}$), we get

$$\int d\theta\, f(\theta) = \frac{\partial f(\theta)}{\partial \theta} = \frac{\partial f(\theta'/a)}{\partial \theta'/a} = a\int d\theta'\, f(\theta'/a)$$

$$\longrightarrow \quad d\theta' = \frac{1}{a}\, d\theta$$

for n variables we get under $\theta_i \mapsto \theta_i' = a_{ij}\theta_j$

$$\int d\theta_1 \cdots d\theta_n\, f(\theta) = \frac{\partial}{\partial \theta_1} \cdots \frac{\partial}{\partial \theta_n} f(\theta)$$

$$= \sum_{k_i=1}^{n} \frac{\partial \theta'_{k_1}}{\partial \theta_1} \cdots \frac{\partial \theta'_{k_n}}{\partial \theta_n} \frac{\partial}{\partial \theta'_{k_1}} \cdots \frac{\partial}{\partial \theta'_{k_n}} f(a^{-1}\theta')$$

$$= \sum_{k_i=1}^{n} a_{k_1 1} \cdots a_{k_n n} \frac{\partial}{\partial \theta'_{k_1}} \cdots \frac{\partial}{\partial \theta'_{k_n}} f(a^{-1}\theta')$$

$$= \sum_{k_i=1}^{n} \varepsilon_{k_1 \cdots k_n} a_{k_1 1} \cdots a_{k_n n} \frac{\partial}{\partial \theta_1'} \cdots \frac{\partial}{\partial \theta_n'} f(a^{-1}\theta')$$

$$= \det a \int d\theta_1' \cdots d\theta_n'\, f(a^{-1}\theta')$$

$\longrightarrow$ thus the integral measure transforms as:

$$d\theta_1 d\theta_2 \cdots d\theta_n = \det a\, d\theta_1' d\theta_2' \cdots d\theta_n'$$

Gaussian integral

Let us consider the integral

$$I = \int d\theta_1^* d\theta_1 \dots d\theta_n^* d\theta_n \, e^{-\sum_{i,j} \theta_i^* M_{ij} \theta_j}$$

where $\{\theta_i\}$ and $\{\theta_i^*\}$ are two sets of independent Grassmann variables

→ the $n \times n$ c-number matrix M is taken to be anti-symmetric since θ_i and θ_i^* anti-commute

Performing the change of variables

$$\theta_i' = \sum_j M_{ij} \theta_j ,$$

gives

$$I = \det M \int d\theta_1^* d\theta_1' \dots d\theta_n^* d\theta_n' e^{-\sum_i \theta_i^* \theta_i'}$$

$$= \det M \left[\int d\theta^* d\theta (1 + \theta' \theta) \right]^n$$

$$= \det M$$

Complex conjugation:

Let $\{\theta_i\}$ and $\{\theta_i^*\}$ be two sets of Grassmann number generators. Define complex conjugation of θ_i by

$$(\theta_i)^* := \theta_i^* \quad \text{and} \quad (\theta_i^*)^* := \theta_i$$

and

$$(\theta_i \theta_j)^* = \theta_j^* \theta_i^*$$

$\rightarrow$ real c-number $\theta_i \theta_i^*$ satisfies reality condition:

$$(\theta_i \theta_i^*)^* = \theta_i \theta_i^*$$

Coherent states for fermions

Let us consider fermionic annihilation and creation operators c, $c^\dagger$ satisfying

$$\{c, c\} = \{c^\dagger, c^\dagger\} = 0, \quad \{c, c^\dagger\} = 1$$

with number operator $N = c^\dagger c$

$\rightarrow$ has eigenstates $|0\rangle$ and $|1\rangle$

$$\mathcal{H} = \text{span}\{|0\rangle, |1\rangle\}$$

An arbitrary vector $|f\rangle$ in $\mathcal{H}$ can be written as

$$|f\rangle = |0\rangle f_0 + |1\rangle f_1 ,$$

where $f_0, f_1 \in \mathbb{C}$.

Now consider the states

$$|\theta\rangle = |0\rangle + |1\rangle \theta \qquad (1)$$
$$\langle\theta| = \langle 0| + \theta^* \langle 1|$$

where θ and θ^* are Grassmann numbers.

$\longrightarrow$ states in (1) are "coherent states"

we have: $c|\theta\rangle = |0\rangle \theta = |\theta\rangle \theta$

$\langle\theta| c^\dagger = \theta^* \langle 0| = \theta^* \langle\theta|$

We have the following identities:

$$\langle\theta'|\theta\rangle = 1 + \theta'^* \theta = e^{\theta'^* \theta}$$
$$\langle\theta|f\rangle = f_0 + \theta^* f_1 \qquad \text{(exercise)}$$
$$\langle\theta|c^\dagger|f\rangle = \langle\theta|1\rangle f_0 = \theta^* f_0 = \theta^* \langle\theta|f\rangle$$
$$\langle\theta|c|f\rangle = \langle\theta|0\rangle f_1 = \frac{\partial}{\partial\theta^*} \langle\theta|f\rangle$$

Let $h(c, c^\dagger) = h_{00} + h_{10} c^\dagger + h_{01} c + h_{11} c^\dagger c, \quad h_{ij} \in \mathbb{C}$

be an arbitrary function of c and $c^\dagger$

$\longrightarrow \langle 0|h|0\rangle = h_{00}, \quad \langle 0|h|1\rangle = h_{01},$

$\langle 1|h|0\rangle = h_{10}, \quad \langle 1|h|1\rangle = h_{00} + h_{11}$

$\longrightarrow \langle \theta|h|\theta'\rangle = (h_{00} + \theta^* h_{10} + h_{01}\theta' + \theta^*\theta' h_{11}) e^{\theta^*\theta'}$

Lemma 1:

Let $|\theta\rangle$ and $\langle\theta|$ be defined as before.

$\longrightarrow$ completeness relation:

$$\int d\theta^* d\theta \, |\theta\rangle\langle\theta| e^{-\theta^*\theta} = \mathbb{1}$$

Proof:

$$\int d\theta^* d\theta \, |\theta\rangle\langle\theta| e^{-\theta^*\theta}$$

$$= \int d\theta^* d\theta \, (|0\rangle + |1\rangle\theta)(\langle 0| + \theta^*\langle 1|)(1 - \theta^*\theta)$$

$$= \int d\theta^* d\theta \, (|0\rangle\langle 0| + |1\rangle\theta\langle 0| + |0\rangle\theta^*\langle 1| + |1\rangle\theta\theta^*\langle 1|)(1 - \theta^*\theta)$$

$$= |0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{1}$$

$\square$

Partition function of a fermionic oscillator

Given the Hamiltonian

$$H = \left(c^{\dagger}c - \tfrac{1}{2}\right)\omega,$$

with eigenvalues $\pm\frac{\omega}{2}$, the partition function is defined as

$$\begin{aligned} Z(\beta) &= \operatorname{Tr} e^{-\beta H} \\ &= \sum_{n=0}^{1} \langle n| e^{-\beta H} |n\rangle \\ &= e^{\beta\omega/2} + e^{-\beta\omega/2} = 2\cosh(\beta\omega/2) \end{aligned}$$

Lemma 2:

Let H be given as above. Then

$$\operatorname{Tr} e^{-\beta H} = \int d\theta^* d\theta \langle -\theta| e^{-\beta H} |\theta\rangle e^{-\theta^*\theta}$$